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THE ASYMPTOTIC EXPANSION OF ELASTIC FIELDS IN THE VICINITY OF THE CONTOUR OF A PLANE CRACK AT THE INTERFACE OF TWO MATERIALS†

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The stress–strain state in the neighbourhood of the front of a plane crack at the interface of two dissimilar half-spaces of ideally elastic isotropic materials is investigated. The form of the asymptotic expansions of the projections of the displacement vector onto the axis, directed along the tangent, the principal normal and the binormal to the crack contour is obtained. It is shown that asymptotic expansions of the projections of the displacement vector onto directions corresponding to the tangent and principal normal, beginning with the second-order term of the expansion, include both terms with half-integer and complex powers of the distances to the crack contour. This indicates that these projections of the solutions of the three-dimensional problem have singularities, defined by the solutions of both the antiplane and plane strain problems of cracks at the interface of materials. The singularities of the projection of the displacement vector on to the binormal correspond to the singularities of the solution of the plane strain problem.

1. FORMULATION OF THE PROBLEM

Suppose a crack occupies the region G in the $z = 0$ plane of an unbounded elastic space. We will assume that the boundary ∂G of the region G is an infinitely smooth curve. Poisson's ratio and the shear modulus are equal to ν_1 and μ_1 for the upper half-space ($z > 0$) and ν_2 and μ_2 for the lower half-space ($z < 0$), respectively. In the $z = 0$ plane, outside the crack G, it is assumed that the conditions of strong adhesion between the half-planes are satisfied, i.e.

$$u^{(1)}(x,y,0) = u^{(2)}(x,y,0), \quad \sigma_{3j}^{(1)}(x,y,0) = \sigma_{3j}^{(2)}(x,y,0)$$
$$j = 1, 2, 3; \ (x, y) \notin G$$

Here $u^{(k)}(x, y, 0)$ ($k = 1, 2$) are the displacement vectors and $\sigma_{3j}^{(k)}(x, y, 0)$ are the components of the stress tensor; the superscript (1) denotes the upper half-space and the superscript (2) the lower half-space.

In the neighbourhood of an arbitrary point $(x', y', 0) \in \partial G$ we will introduce a local system of coordinates, defined by the directions of the tangent, the principal normal and the binormal to ∂G at this point. The components of the displacement vector in these directions will be denoted by $u_{\text{tan}}^{(k)}, u_{\text{nor}}^{(k)}, u_3^{(k)}$, respectively. As is well known, linear fracture mechanics is based on an analysis of the principal terms of the asymptotic expansions of the singular components of these components of the displacements in the neighbourhood of the crack front. Nevertheless, in many cases it is useful to know not only the principal terms of the expansion but also the form of the whole asymptotic series, or at least some of its next terms. The need for such additional information arises, for example, when constructing a closed system of formulae of the variation of the solution of the problem, due to the variation of the crack contour (see [1]), when refining numerical solutions in the neighbourhood of the crack front and when constructing refined fracture criteria. Hence the purpose of this paper is to investigate the complete form of the asymptotic expansion of the components of the displacements $u_{\text{tan}}^{(k)}, u_{\text{nor}}^{(k)}, u_3^{(k)}$ in the vicinity of the point $(x', y', 0)$ when the point (x, y, z) considered are situated in the plane that passes through the point $(x', y', 0)$ and is normal to ∂G at this point.

At first glance it might seem that the solution of this problem is fairly simple, since the expansion of the solutions of the antiplane and plane strain problems of interface cracks are well known, while the forms of the expansions of $u_{\text{tan}}^{(k)}$ and $(u_{\text{nor}}^{(k)}, u_3^{(k)})$ should be identical, as can be assumed, with the forms

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of the expansions of the corresponding antiplane and plane strain problems. However, this assumption is incorrect.

As is well known, the solution of the antiplane problem of an interface crack in the vicinity of its tip can be represented in the form (henceforth summation is everywhere carried out from $n = 0$ to $n = \infty$)

$$u_{\text{ant}}^{(k)} = \sum r^{n+\frac{1}{2}} f_n^{*(k)}(\varphi) + \sum r^n h_n^{*(k)}(\varphi), \quad k = 1,2 \tag{1.1}$$

Here r is the distance to the crack tip and φ is the polar angle, where $\varphi = \pi$ corresponds to the upper edge of the crack, $\varphi = -\pi$ corresponds to the lower edge and $\varphi = 0$ is the line along which materials 1 and 2 are joined.

The solution of the plane strain problem can be represented in the form

$$u_{\text{pl}1}^{(k)} + iu_{\text{pl}2}^{(k)} = \sum r^{n+\frac{1}{2}+i\varepsilon} f_n^{(k)}(\varphi) + \sum r^n h_n^{(k)}(\varphi) \tag{1.2}$$

$$\varepsilon = \frac{1}{2\pi} \ln \frac{\mu_2 \varkappa_1 + \mu_1}{\mu_1 \varkappa_2 + \mu_2}, \quad \varkappa_j = 3 - 4\nu_j, \quad j = 1,2$$

where $u_{\text{pl}1}^{(k)}$ and $u_{\text{pl}2}^{(k)}$ are displacements perpendicular to the line of the crack and along it and $f_n^{(k)}(\varphi)$, $h_n^{(k)}(\varphi)$ are complex-valued functions.

We will show below that although the form of the principal term of the singular component of the asymptotic expansion of $u_{\text{tan}}^{(k)}$ is identical in form with the principal term of expansion (1.1), in fact for weaker singularities in the three-dimensional problem, due to the dependence of the solution on the variable tangential to the crack contour, interaction of singularities of types (1.1) and (1.2) occurs, as a result of which the expansion of $u_{\text{tan}}^{(k)}$ includes both terms of the order of $r^{n+\frac{1}{2}}$ and terms of the order of $r^{n+\frac{3}{2}+i\varepsilon}$ ($n = 0, 1, 2, \ldots$). Here r is the distance to the crack contour.

In exactly the same way the asymptotic series for the singular component of the displacement $u_{\text{nor}}^{(k)}$ begins with a term of the order of $r^{\frac{1}{2}+i\varepsilon}$, which corresponds to the solution of plane strain problem (1.2), and moreover contains both terms of the order of $r^{n+\frac{3}{2}+i\varepsilon}$ and terms of the order of $r^{n+\frac{3}{2}}$ ($n = 0, 1, 2, \ldots$).

There are two fundamental approaches to solving the problem of the form of the asymptotic expansion of the displacements and stresses in the vicinity of the crack contour. One of these is based on the construction of an analytical solution of some crack problem and its subsequent asymptotic expansion. Here it is necessary that the given particular solution should include the main properties of the general solution of the problem.

The other approach is to consider the crack problem in the form of a half-plane. All the components of the displacement vector are represented in the form $r^{\lambda}\Phi_j(\varphi, \tau)$, where τ is a coordinate on the axis, directed along the crack front, (r, φ, τ) are cylindrical coordinates and j indicates the component of the displacements. A solution of the equations of the theory of elasticity is sought which satisfies the conditions of strong adhesion at the interface of the materials and which gives zero loads on the crack surfaces.

The first approach is difficult to use because there is an extremely limited number of analytically solved three-dimensional problems of interface cracks. Among these is the problem of a penny-shaped crack, to the surfaces of which uniform normal loads are applied [2, 3]. Expansions of the normal and radial displacements in this problem have the form (1.2) and do not contain half-integer powers of the distances to the periphery bounding the crack. However, since this problem is axisymmetric and the displacements in the tangential direction are zero, interaction of singularities of types (1.1) and (1.2) obviously cannot occur. A similar situation also arises for solutions of the problem of a penny-shaped crack, to the surfaces of which axisymmetric radial shear loads are applied [4].

There is also an analytical solution of another problem – a penny-shaped crack subject to the torsion loads [4]. In this problem the tangential component of the displacement vector is given by an asymptotic expansion of the form (1.1), but this stressed state is also not sufficiently general, since the radial displacement and the displacement normal to the plane of the crack in this case are zero.

A solution of the problem of a penny-shaped crack, to the surfaces of which loads of a fairly general form are applied, was obtained in [5]. However, it is quite difficult to obtain from this the complete form of the asymptotic expansion in the vicinity of the crack front, since this solution is represented in a fairly complex form in terms of a Radon transformation.

In view of all this and in order to carry out a clearer investigation of the reasons why interaction of singularities of types (1.1) and (1.2) occur, we will use the second approach below to obtain the form of the asymptotic expansion of the solution. It should be noted here that if all the components of the displacement vector are sought in the form $r^{\lambda}{}_j\Phi_j(\varphi, \tau)$ with the same exponent λ, nothing, apart from solutions of the plane and antiplane problems, can be obtained. The interaction of the singularities can be established if we represent the components of the displacements in the form $r^{\lambda}{}_j\Phi_j(\varphi, \tau)$ with different exponents λ_j, in which case, as will be shown below, particular solutions of this type are obtained by slight modifications of the solutions of the antiplane and plane problems.

2. SOLUTION OF THE ANTIPLANE PROBLEM AND ITS MODIFICATION

Suppose the crack is in the $z = 0$ plane and has the form of a half-plane occupying the region $x < 0$. We will denote the displacements in the x, y and z directions by $u_1^{(k)}$, $u_2^{(k)}$ and $u_3^{(k)}$ respectively, where the superscript (k) indicates, as above, the half-space being considered. Note that for this crack the displacements $u_1^{(k)}$ are identical with $u_{\text{nor}}^{(k)}$ and the displacements $u_2^{(k)}$ are identical with $u_{\tan}^{(k)}$.

The problem of finding solutions of the equations of the theory of elasticity, which satisfy the homogeneous boundary conditions, is formulated as follows. The displacements satisfy Lame's equations

$$\Delta u_j^{(k)} + \frac{1}{1-2\nu_k}\frac{\partial\theta^{(k)}}{\partial x_j} = 0, \quad \theta^{(k)} = \frac{\partial u_1^{(k)}}{\partial x} + \frac{\partial u_2^{(k)}}{\partial y} + \frac{\partial u_3^{(k)}}{\partial z} \tag{2.1}$$

(x_j for $j = 1, 2, 3$ correspond to x, y, z). We will assume that the functions $u_j^{(1)}(x, y, z)$ are defined in the half-space $z > 0$, and the function $u_j^{(2)}(x, y, z)$ is defined in the half-space $z < 0$. Equations (2.1) are then satisfied for $k = 1$ in the half-space $z > 0$ and for $k = 2$ in the half-space $z < 0$.

The conditions for there to be no load on the crack surface can be written in the form

$$\sigma_{31}^{(k)}(x,y,0) = \sigma_{32}^{(k)}(x,y,0) = \sigma_{33}^{(k)}(x,y,0) = 0 \quad \text{for} \quad x<0 \tag{2.2}$$

where $\sigma_{3j}^{(k)}$ $(j = 1, 2, 3)$ are the components of the stress tensor.

The conditions for strong adhesion between the half-spaces outside the crack have the form

$$u_j^{(1)}(x,y,0) = u_j^{(2)}(x,y,0), \quad \sigma_{3j}^{(1)}(x,y,0) = \sigma_{3j}^{(2)}(x,y,0) \tag{2.3}$$

for $x > 0, j = 1, 2, 3$.

We briefly recall the solution of the antiplane problem, which is obtained if we put

$$u_1^{(k)} = u_3^{(k)} = 0, \quad u_2^{(k)} = \overset{0}{u}{}_2^{(k)}(x,z)$$

We change to polar coordinates

$$x = r\cos\varphi, \qquad z = r\sin\varphi, \qquad -\pi < \varphi < \pi$$

In these coordinates the upper half-space is defined by the conditions $0 < \varphi < \pi$ and the lower half-space by the conditions $-\pi < \varphi < 0$. The crack surfaces correspond to $\varphi = \pm\pi$ and the interface of the materials corresponds to $\varphi = 0$.

We will seek a solution in the form

$$\overset{0}{u}{}_2^{(1)}(r,\varphi) = r^{\lambda}f_1(\varphi), \quad 0 \le \varphi \le \pi; \quad \overset{0}{u}{}_2^{(2)}(r,\varphi) = r^{\lambda}f_2(\varphi), \quad -\pi \le \varphi \le 0$$

It then follows from Eq. (2.1) for $j = 2$ that

$$f_k(\varphi) = c_{k1}\cos\lambda\varphi + c_{k2}\sin\lambda\varphi$$

where c_{k1} and c_{k2} are constants.

Substituting these expressions into conditions (2.2) and (2.3) we obtain a system of four linear homogeneous equations

$$\sigma_{32}^{(1)}(r,\pi)=0,\quad \sigma_{32}^{(2)}(r,-\pi)=0$$

$$\overset{0}{u}{}_2^{(1)}(r,0)=\overset{0}{u}{}_2^{(2)}(r,0),\quad \sigma_{32}^{(1)}(r,0)=\sigma_{32}^{(2)}(r,0)$$

in the four unknowns c_{k1} and c_{k2} ($k = 1, 2$).

In order for a non-zero solution of this system to exist its determinant must equal zero. From this condition we obtain the equality $\sin\lambda\pi\cos\lambda\pi = 0$, the roots of which are $\lambda = n$ and $\lambda = n + 1/2$, where n are integers. From the condition that the energy must be finite it follows that $n = 0, 1, 2, \ldots$. The roots $\lambda = n$ correspond to the regular part of the expansion of the solution of the antiplane problem, while the roots $\lambda = n + 1/2$ correspond to the singular part. The eigenfunctions $f_1(\varphi)$ and $f_2(\varphi)$, corresponding to the value $\lambda = n + 1/2$, have the form

$$f_1(\varphi)=C_n\sin(n+\tfrac{1}{2})\varphi,\quad f_2(\varphi)=C_n(\mu_1/\mu_2)\sin(n+\tfrac{1}{2})\varphi$$

where C_n is an arbitrary constant.

Hence, the singular solutions of the antiplane problem of interest here have the form

$$\begin{gathered} u_1^{(k)}=u_3^{(k)}=0 \\ \overset{0}{u}{}_{n2}^{(1)}=C_n r^{n+1/2}\sin(n+\tfrac{1}{2})\varphi,\quad \overset{0}{u}{}_{n2}^{(2)}=C_n(\mu_1/\mu_2)r^{n+1/2}\sin(n+\tfrac{1}{2})\varphi \end{gathered} \tag{2.4}$$

The subscript n for the displacements indicates which singular solution is in fact taken.

We will now modify solutions (2.4) by constructing singular solutions of problem (2.1)–(2.3), in which the orders of the singularities of the displacements $u_1^{(k)}$ will be equal to $r^{n+3/2}$ ($n = 0, 1, 2, \ldots$).

We will seek solutions of problem (2.1)–(2.3) in the form

$$u_n^{(1)}(r,\varphi,y)=(u_{n1}^{(1)}(r,\varphi),y\overset{0}{u}{}_{n2}^{(1)},0),\quad u_n^{(2)}(r,\varphi,y)=(u_{n1}^{(2)}(r,\varphi),y\overset{0}{u}{}_{n2}^{(2)},0)$$

We will choose the displacements $u_{n1}^{(1)}$ and $u_{n1}^{(2)}$ so that $\theta^{(1)}$ and $\theta^{(2)}$, as in the antiplane problem, remain equal to zero. In order to do this the conditions $\partial u_{n1}^{(k)}/\partial x + \overset{0}{u}{}_{n2}^{(k)} = 0$ must be satisfied, or in polar coordinates

$$\cos\varphi\frac{\partial u_{n1}^{(k)}}{\partial r}-\frac{\sin\varphi}{r}\frac{\partial u_{n1}^{(k)}}{\partial\varphi}+\overset{0}{u}{}_{n2}^{(k)}=0 \tag{2.5}$$

Choosing the solutions of Eq. (2.5) in the form $u_{n1}^{(k)}/r^{\gamma}g_k(\varphi)$, it can be shown that the solutions are the functions

$$\begin{aligned} u_{n1}^{(1)}&=-C_n(n+\tfrac{3}{2})^{-1}r^{n+3/2}\sin(n+\tfrac{3}{2})\varphi \\ u_{n1}^{(2)}&=-C_n(n+\tfrac{3}{2})^{-1}(\mu_1/\mu_2)r^{n+3/2}\sin(n+\tfrac{3}{2})\varphi \end{aligned} \tag{2.6}$$

Since $\theta^{(k)} = 0$, and $\overset{0}{u}{}_{n2}^{(k)}(r, \varphi)$ and $u_{n1}^{(k)}(r, \varphi)$ are harmonic functions, Eqs (2.1) are obviously satisfied. The satisfaction of conditions (2.2) and (2.3) can also be verified without difficulty.

Hence it can be shown that the displacements

$$u_n^{(1)}=(u_{n1}^{(1)},y\overset{0}{u}{}_{n2}^{(1)},0),\quad u_n^{(2)}=(u_{n1}^{(2)},y\overset{0}{u}{}_{n2}^{(2)},0)$$

where $u_{n1}^{(k)}$ are defined in (2.6) while $\overset{0}{u}{}_{n2}^{(k)}$, defined in (2.4), are the solutions of boundary-value problem (2.1)–(2.3). Hence it follows that in the expansion of the displacements $u_1^{(k)} = u_{\text{nor}}^{(k)}$ for arbitrary smooth loads terms of the order of $r^{n+3/2}$ ($n = 0, 1, 2, \ldots$) must be present.

Remark. One cannot construct a solution of problem (2.1)–(2.3), in which $u_{n1}^{(k)} = 0, u_{n3}^{(k)} = u_{n3}^{(k)}(r, \varphi)$, in the same way. In fact if we seek a solution in the form

$$u_n^{(k)}(r,\varphi,y)=(0,y\overset{0}{u}{}_{n2}^{(k)},u_{n3}^{(k)}(r,\varphi))$$

and chose the displacements $u_{n3}^{(k)}$ so that $\theta^{(k)} = 0$, the following equalities must be satisfied

$$\partial u_{n3}^{(k)}/\partial z + \overset{0}{u}{}_{n2}^{(k)} = 0, \quad k = 1,2 \tag{2.7}$$

By virtue of the fact that $\theta^{(k)} = 0$, the stresses $\sigma_{33}^{(k)}$ can be written in the form $\sigma_{33}^{(k)} = 2\mu_k \partial u_{n3}^{(k)}/\partial z$. Hence and also from (2.7) it follows that $\sigma_{33}^{(k)} = -2\mu_k \overset{0}{u}{}_{n2}^{(k)}$. Since $\overset{0}{u}{}_{n2}^{(1)}(r, \pi)$ and $\overset{0}{u}{}_{n2}^{(2)}(r, -\pi)$ are non-zero, it is impossible to satisfy the boundary conditions on the crack surface ($\sigma_{33}^{(1)}(r, \pi) = 0$, $\sigma_{33}^{(2)}(r, -\pi) = 0$).

This remark, of course, does not prove that there are no half-integer powers of r in the expansion of the displacements $u_3^{(k)}$. Nevertheless, an analysis of the solution of the problem of a penny-shaped crack at the interface of half-spaces, obtained previously [5], also shows that the expansion of $u_3^{(k)}$ does not contain terms of order $r^{n+3/2}$.

3. THE SOLUTION OF THE PLANE PROBLEM AND ITS MODIFICATION

We will consider the same problem of a crack having the form of a half-plane as in Section 2. It is more convenient in this case to change to a cylindrical system of coordinates r, φ, y ($x = r\cos\varphi$, $z = r\sin\varphi$). Lamé's equations in a cylindrical system of coordinates take the form

$$\begin{gathered}\left(\Delta - \frac{1}{r^2}\right)u_r^{(k)} - \frac{2}{r^2}\frac{\partial u_\varphi^{(k)}}{\partial\varphi} + \frac{1}{1-2\nu_k}\frac{\partial\theta^{(k)}}{\partial r} = 0\\ \left(\Delta - \frac{1}{r^2}\right)u_\varphi^{(k)} + \frac{2}{r^2}\frac{\partial u_r^{(k)}}{\partial\varphi} + \frac{1}{1-2\nu_k}\frac{1}{r}\frac{\partial\theta^{(k)}}{\partial\varphi} = 0\\ \Delta u_2^{(k)} + \frac{1}{1-2\nu_k}\frac{\partial\theta^{(k)}}{\partial y} = 0\end{gathered} \tag{3.1}$$

The components of the stress tensor occurring in the boundary conditions can be expressed in terms of the displacements as follows:

$$\begin{gathered}\tau_{\varphi r}^{(k)} = \mu_k\left(\frac{1}{r}\frac{\partial u_r^{(k)}}{\partial\varphi} + \frac{\partial u_\varphi^{(k)}}{\partial r} - \frac{u_\varphi^{(k)}}{r}\right)\\ \tau_{\varphi y}^{(k)} = \mu_k\left(\frac{\partial u_\varphi^{(k)}}{\partial y} + \frac{1}{r}\frac{\partial u_2^{(k)}}{\partial\varphi}\right)\\ \sigma_\varphi^{(k)} = 2\mu_k\left[\frac{\nu_k}{1-2\nu_k}\left(\frac{\partial u_2^{(k)}}{\partial y} + \frac{\partial u_r^{(k)}}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi^{(k)}}{\partial\varphi} + \frac{u_r^{(k)}}{r}\right) + \frac{1}{r}\frac{\partial u_\varphi^{(k)}}{\partial\varphi} + \frac{u_r^{(k)}}{r}\right]\end{gathered} \tag{3.2}$$

The conditions for there to be no loads on the crack surfaces can be written in the form

$$\begin{gathered}\tau_{\varphi r}^{(1)}(r,\pi,y) = 0, \quad \tau_{\varphi y}^{(1)}(r,\pi,y) = 0, \quad \sigma_\varphi^{(1)}(r,\pi,y) = 0\\ \tau_{\varphi r}^{(2)}(r,-\pi,y) = 0, \quad \tau_{\varphi y}^{(2)}(r,-\pi,y) = 0, \quad \sigma_\varphi^{(2)}(r,-\pi,y) = 0\end{gathered} \tag{3.3}$$

Finally, the conditions for strong adhesion between the half-spaces outside the crack take the form

$$\begin{gathered}u_r^{(1)}(r,0,y) = u_r^{(2)}(r,0,y), \quad u_\varphi^{(1)}(r,0,y) = u_\varphi^{(2)}(r,0,y)\\ u_2^{(1)}(r,0,y) = u_2^{(2)}(r,0,y), \quad \tau_{\varphi r}^{(1)}(r,0,y) = \tau_{\varphi r}^{(2)}(r,0,y)\\ \tau_{\varphi y}^{(1)}(r,0,y) = \tau_{\varphi y}^{(2)}(r,0,y), \quad \sigma_\varphi^{(1)}(r,0,y) = \sigma_\varphi^{(2)}(r,0,y)\end{gathered} \tag{3.4}$$

The solution of the plane problem is obtained if we put

$$u_r^{(k)} = \overset{0}{u}{}_r^{(k)}(r,\varphi), \quad u_\varphi^{(k)} = \overset{0}{u}{}_\varphi^{(k)}(r,\varphi), \quad u_2^{(k)} = 0$$

The singularities of the solution of the plane problem were investigated for the first time in [6], where the stress functions $U_1(r, \varphi)$ in the upper half-plane ($0 \leqslant \varphi \leqslant \pi$) and $U_2(r, \varphi)$ in the lower half-plane ($-\pi \leqslant \varphi \leqslant 0$) were considered. These were represented in the form $U_k(r, \varphi) = r^{\lambda+1}F_k(\varphi)$ ($k = 1, 2$). Since the stress function must satisfy the biharmonic equation $\Delta\Delta U_k = 0$, the functions $F_k(\varphi)$ have the form

$$F_k(\varphi) = a_k \sin(\lambda+1)\varphi + b_k \cos(\lambda+1)\varphi + c_k \sin(\lambda-1)\varphi + d_k \cos(\lambda-1)\varphi$$

where a_k, b_k, c_k, d_k are constants.

The components of the stress tensor can be expressed in terms of the stress function as follows:

$$\sigma_r^{(k)} = r^{\lambda-1}[F_k''(\varphi) + (\lambda+1)F_k(\varphi)], \quad \sigma_\varphi^{(k)} = r^{\lambda-1}\lambda(\lambda+1)F_k(\varphi)$$
$$\tau_{\varphi r}^{(k)} = -\lambda r^{\lambda-1}F_k'(\varphi) \tag{3.5}$$

The components of the displacement vector can be written in the form

$$u_\varphi^{(k)} = \frac{1}{2\mu_k}r^\lambda\{-F_k'(\varphi) - 4(1-\nu_k)[c_k\cos(\lambda-1)\varphi - d_k\sin(\lambda-1)\varphi]\}$$
$$u_r^{(k)} = \frac{1}{2\mu_k}r^\lambda\{-(\lambda+1)F_k(\varphi) + 4(1-\nu_k)[c_k\sin(\lambda-1)\varphi + d_k\cos(\lambda-1)\varphi]\} \tag{3.6}$$

The conditions for there to be no loads on the crack surfaces (3.3), by virtue of expressions (3.5) for the stresses, can be written in the form

$$F_1(\pi) = F_1'(\pi) = F_2(-\pi) = F_2'(-\pi) = 0 \tag{3.7}$$

The conditions for the forces-outside the crack (3.4) to be equal, from expressions (3.5) have the form

$$F_1(0) = F_2(0), \quad F_1'(0) = F_2'(0) \tag{3.8}$$

The conditions for the displacements outside the crack (3.8) to be equal, taking (3.6) into account, reduce to the equations

$$\frac{1}{2\mu_1}[-F_1'(0) - 4(1-\nu_1)c_1] = \frac{1}{2\mu_2}[-F_2'(0) - 4(1-\nu_2)c_2]$$
$$\frac{1}{2\mu_1}[-(\lambda+1)F_1(0) + 4(1-\nu_1)d_1] = \frac{1}{2\mu_2}[-(\lambda+1)F_2(0) + 4(1-\nu_2)d_2] \tag{3.9}$$

Equations (3.7)–(3.9) represent eight linear homogeneous equations in the eight unknowns a_k, b_k, c_k, d_k ($k = 1, 2$), which define the functions $F_k(\varphi)$. In order for a non-zero solution of this system to exist, its determinant must be equal to zero. From this condition it was found [6] that $\lambda = n + 1/2 + i\varepsilon$ (the quantity ε is defined in (1.2)). These values of λ only define the singular component of expansion (1.2).

Using the techniques of the theory of functions of a complex variable it was established in [7] that the expansion of the solution of the plane problem has the form (1.2).

The results obtained in [6] were refined in [8] by a more accurate calculation of the determinant of system of equations (3.7)–(3.9). In particular, it was established that the equation obtained by equating the determinant of the system to zero, in addition to complex roots, also has roots $\lambda = n$. All the values of λ corresponding to expansion (1.2), obtained in [7], were derived by the same methods as in [6]. In addition, the coefficients a_k, b_k, c_k, d_k, defining the eigenfunctions $F_k(\varphi)$, were calculated in [8].

We will denote the solution of the plane problem corresponding to $\lambda = n + 1/2 + i\varepsilon$ by $\overset{0}{u}{}_{nr}^{(k)}(r, \varphi)$, $\overset{0}{u}{}_{n\varphi}^{(k)}(r, \varphi)$, and the eigenfunctions corresponding to this root by $F_{nk}(\varphi)$. By the results obtained previously [8]

$$a_{n1} = iM_n \frac{e^{\pi\varepsilon} + (n + \frac{1}{2} + i\varepsilon)e^{-\pi\varepsilon}}{n + \frac{3}{2} + i\varepsilon}, \quad b_{n1} = M_n \frac{e^{\pi\varepsilon} - (n + \frac{1}{2} + i\varepsilon)e^{-\pi\varepsilon}}{n + \frac{3}{2} + i\varepsilon}$$
$$c_{n1} = -iM_n e^{-\pi\varepsilon}, \quad d_{n1} = M_n e^{-\pi\varepsilon}, \quad a_{n2} = iM_n \frac{e^{-\pi\varepsilon} + (n + \frac{1}{2} + i\varepsilon)e^{\pi\varepsilon}}{n + \frac{3}{2} + i\varepsilon} \tag{3.10}$$
$$b_{n2} = M_n \frac{e^{-\pi\varepsilon} - (n + \frac{1}{2} + i\varepsilon)e^{\pi\varepsilon}}{n + \frac{3}{2} + i\varepsilon}, \quad c_{n2} = -iM_n e^{\pi\varepsilon}, \quad d_{n2} = M_n e^{\pi\varepsilon}$$

The subscript n indicates to which eigenfunctions the constants correspond, and M_n are arbitrary constants.

We will now modify the solution of the plane problem by constructing the singular solutions of problem (3.1)–(3.3), in which the orders of the singularities of the displacement $u_2^{(k)}$ are equal to $r^{n+3/2+i\varepsilon}$ $(n = 0, 1, 2, \ldots)$. We will seek solutions of problem (3.1)–(3.3) in the form

$$u_n^{(1)}(r,\varphi,y) = (u_{nr}^{(1)}, u_{n\varphi}^{(1)}, u_{n2}^{(1)}) = (y\overset{0}{u}{}_{nr}^{(1)}(r,\varphi),\ y\overset{0}{u}{}_{n\varphi}^{(1)}(r,\varphi),\ u_{n2}^{(1)}(r,\varphi))$$

$$u_n^{(2)}(r,\varphi,y) = (u_{nr}^{(2)}, u_{n\varphi}^{(2)}, u_{n2}^{(2)}) = (y\overset{0}{u}{}_{nr}^{(2)}(r,\varphi),\ y\overset{0}{u}{}_{n\varphi}^{(2)}(r,\varphi),\ u_{n2}^{(2)}(r,\varphi))$$

It can be shown that for the chosen form of the displacements $u_n^{(k)}(r, \varphi, y)$ the equality $\theta^{(k)} = y\overset{0}{\theta}{}_n^{(k)}$ holds, where $\overset{0}{\theta}{}_n^{(k)}$ is the value of the first invariant of the strain tensor for the corresponding plane problem. The following equalities also hold

$$\left(\Delta - \frac{1}{r^2}\right)u_{nr}^{(k)} = y\left(\Delta - \frac{1}{r^2}\right)\overset{0}{u}{}_{nr}^{(k)}, \quad \left(\Delta - \frac{1}{r^2}\right)u_{n\varphi}^{(k)} = y\left(\Delta - \frac{1}{r^2}\right)\overset{0}{u}{}_{n\varphi}^{(k)}$$

$$\frac{\partial u_{nr}^{(k)}}{\partial\varphi} = y\frac{\partial \overset{0}{u}{}_{nr}^{(k)}}{\partial\varphi}, \quad \frac{\partial u_{n\varphi}^{(k)}}{\partial\varphi} = y\frac{\partial \overset{0}{u}{}_{n\varphi}^{(k)}}{\partial\varphi}$$

Hence it follows that the first and second of Eqs (3.1) will be satisfied.

It follows from the first and third expressions of (3.2) that $\tau_{\varphi r}^{(k)} = y\overset{0}{\tau}{}_{\varphi r}^{(k)}$ and $\sigma_\varphi^{(k)} = y\overset{0}{\sigma}{}_\varphi^{(k)}$, where $\overset{0}{\tau}{}_{\varphi r}^{(k)}$ and $\overset{0}{\sigma}{}_\varphi^{(k)}$ are the stresses in the corresponding plane problem. Hence it follows that the first, third, fourth and sixth equations of (3.3) are satisfied, and also the fourth and sixth equations of (3.4). In addition it is obvious that the first and second equalities of (3.4) are satisfied.

Consequently, to solve problem (3.1)–(3.3) we need to chose the functions $u_{n2}^{(k)}(r, \varphi)$ so as to satisfy the third of equations (3.1) and also the boundary conditions

$$\tau_{\varphi y}^{(1)}(r,\pi,y) = \tau_{\varphi y}^{(2)}(r,-\pi,y) = 0, \quad u_{n2}^{(1)}(r,0) = u_{n2}^{(2)}(r,0)$$
$$\tau_{\varphi y}^{(1)}(r,\pi,y) = \tau_{\varphi y}^{(2)}(r,0,y) \tag{3.11}$$

Since $\theta^{(k)} = y\overset{0}{\theta}{}_n^{(k)}$, the third of equations (3.1) takes the form

$$\Delta u_{n2}^{(k)}(r,\varphi) = -\frac{1}{1-2\nu_k}\overset{0}{\theta}{}_n^{(k)} \tag{3.12}$$

From expressions (3.5) for $\overset{0}{\sigma}{}_r^{(k)}$ and $\overset{0}{\sigma}{}_\varphi^{(k)}$ and since $2\mu_k\,\overset{0}{\theta}{}_n^{(k)}/(1 - 2\nu_k) = \overset{0}{\sigma}{}_r^{(k)} + \overset{0}{\sigma}{}_\varphi^{(k)}$, we have

$$\overset{0}{\theta}{}_n^{(k)} = \frac{1-2\nu_k}{2\mu_k} r^{n-1/2+i\varepsilon}\left[F_{nk}''(\varphi) + \left(n + \frac{3}{2} + i\varepsilon\right)^2 F_{nk}(\varphi)\right]$$

Now taking into account the form of the functions $F_{nk}(\varphi)$ we obtain

$$\overset{0}{\theta}{}_n^{(k)} = \frac{1-2\nu_k}{2\mu_k} 4\left(n + \frac{1}{2} + i\varepsilon\right) r^{n-1/2+i\varepsilon}\left[c_{nk}\sin\left(n - \frac{1}{2} + i\varepsilon\right)\varphi + d_{nk}\cos\left(n - \frac{1}{2} + i\varepsilon\right)\varphi\right] \tag{3.13}$$

The values of the constants c_{nk} and d_{nk} are given in (3.10).

Substituting (3.13) into (3.12) and seeking a particular solution of (3.12) in the form $u_q^{(k)} = r^{\gamma}\Phi_k(\varphi)$, it can be shown that the following functions are particular solutions

$$u_q^{(k)} = -\frac{1}{2\mu_k} r^{n+3/2+i\varepsilon}\left(c_{nk}\sin\left(n-\frac{1}{2}+i\varepsilon\right)\varphi + d_{nk}\cos\left(n-\frac{1}{2}+i\varepsilon\right)\varphi\right)$$

If we add an arbitrary harmonic function to this particular solution, the new function will also satisfy Eq. (3.12).

Consider solutions of Eq. (3.12) having the form

$$u_{n2}^{(k)} = \frac{1}{2\mu_k} r^{n+3/2+i\varepsilon}\left[iA_{nk}\sin\left(n+\frac{3}{2}+i\varepsilon\right)\varphi + B_{nk}\cos\left(n+\frac{3}{2}+i\varepsilon\right)\varphi - \right.$$
$$\left. - c_{nk}\sin\left(n-\frac{1}{2}+i\varepsilon\right)\varphi - d_{nk}\cos\left(n-\frac{1}{2}+i\varepsilon\right)\varphi\right] \tag{3.14}$$

where A_{nk}, $B_{nk}(k = 1, 2)$ are constants.

Conditions (3.11) represent four equations from which we can determine the four unknown constants A_{nk}, B_{nk} $(k = 1, 2)$. Using (3.14) the condition $u_{n2}^{(1)}(r, 0) = u_{n2}^{(2)}(r, 0)$ leads to the equation

$$\frac{B_{n1}-d_{n1}}{\mu_1} = \frac{B_{n2}-d_{n2}}{\mu_2} \tag{3.15}$$

Taking expression (3.2) for $\tau_{\varphi y}^{(k)}$ into account, we have

$$\tau_{\varphi y}^{(k)} = \mu_k\left(u_{n\varphi}^{0(k)} + \frac{1}{r}\frac{\partial u_{n2}^{(k)}}{\partial\varphi}\right)$$

Hence also from (3.6) and (3.14) we obtain

$$\tau_{\varphi y}^{(k)} = \frac{r^{n+1/2+i\varepsilon}}{2}\left\{-F'_{nk}(\varphi) + \left(n+\frac{3}{2}+i\varepsilon\right)\left(iA_{nk}\cos\left(n+\frac{3}{2}+i\varepsilon\right)\varphi - \right.\right.$$
$$\left. - B_{nk}\sin\left(n+\frac{3}{2}+i\varepsilon\right)\varphi\right) - \left[4(1-\nu_k) + n - \frac{1}{2} + i\varepsilon\right]\times$$
$$\left.\times\left(c_{nk}\cos\left(n-\frac{1}{2}+i\varepsilon\right)\varphi - d_{nk}\sin\left(n-\frac{1}{2}+i\varepsilon\right)\varphi\right)\right\} \tag{3.16}$$

From the condition $\tau_{\varphi y}^{(1)}(r, 0, y) = \tau_{\varphi y}^{(2)}(r, 0, y)$, (3.16) and (3.8) we obtain

$$\left(n+\frac{3}{2}+i\varepsilon\right)iA_{n1} - \left[\varkappa_1 + n + \frac{1}{2} + i\varepsilon\right]c_{n1} = \left(n+\frac{3}{2}+i\varepsilon\right)iA_{n2} - \left[\varkappa_2 + n + \frac{1}{2} + i\varepsilon\right]c_{n2} \tag{3.17}$$

The quantities $\varkappa_1$, $\varkappa_2$ in (3.17) were defined in Section 1.

Substituting the value $\varphi = \pi$ when $k = 1$ and $\varphi = -\pi$ when $k = 2$ into Eq. (3.16) and taking the equalities $\tau_{\varphi y}^{(1)}(r, \pi, y) = 0$, $\tau_{\varphi y}^{(2)}(r, -\pi, y) = 0$ and $F'_{n1}(\pi) = 0$, $F'_{n2}(-\pi) = 0$ into account and also expressions (3.10) we obtain

$$\left(n+\frac{3}{2}+i\varepsilon\right)\left(\frac{A_{n1}+B_{n1}}{2}e^{-\varepsilon\pi}e^{i(n+3/2)\pi} + \frac{A_{n1}-B_{n1}}{2}e^{\varepsilon\pi}e^{-i(n+3/2)\pi}\right) +$$
$$+ M_n\left(\varkappa_1 + n + \frac{1}{2} + i\varepsilon\right)e^{-i(n-1/2)\pi} = 0$$
$$\left(n+\frac{3}{2}+i\varepsilon\right)\left(\frac{A_{n2}-B_{n2}}{2}e^{-\varepsilon\pi}e^{i(n+3/2)\pi} + \frac{A_{n2}+B_{n2}}{2}e^{\varepsilon\pi}e^{-i(n+3/2)\pi}\right) +$$
$$+ M_n\left(\varkappa_2 + n + \frac{1}{2} + i\varepsilon\right)e^{i(n-1/2)\pi} = 0$$

These equations can be rewritten in the simpler form

$$\left(n+\frac{3}{2}+i\varepsilon\right)\left(\frac{A_{n1}+B_{n1}}{2}e^{-\varepsilon\pi}-\frac{A_{n1}-B_{n1}}{2}e^{\varepsilon\pi}\right)-M_n\left(\varkappa_1+n+\frac{1}{2}+i\varepsilon\right)=0 \qquad (3.18)$$

$$\left(n+\frac{3}{2}+i\varepsilon\right)\left(\frac{A_{n2}-B_{n2}}{2}e^{-\varepsilon\pi}-\frac{A_{n2}+B_{n2}}{2}e^{\varepsilon\pi}\right)+M_n\left(\varkappa_2+n+\frac{1}{2}+i\varepsilon\right)=0 \qquad (3.19)$$

Solving system of equations (3.15) and (3.17)–(3.19) we find

$$A_{n1}=\alpha_{n1}e^{-\varepsilon\pi}+\beta_{n1}e^{\varepsilon\pi},\quad B_{n1}=-\alpha_{n1}e^{-\varepsilon\pi}+\beta_{n1}e^{\varepsilon\pi}$$

$$A_{n2}=\alpha_{n2}e^{-\varepsilon\pi}+\beta_{n2}e^{\varepsilon\pi},\quad B_{n2}=\alpha_{n2}e^{-\varepsilon\pi}-\beta_{n2}e^{\varepsilon\pi}$$

where

$$\alpha_{n2}=M_n\frac{(\varkappa_2-1)\mu_1e^{\varepsilon\pi}-(\varkappa_1-1)\mu_2e^{-\varepsilon\pi}}{(n+3/2+i\varepsilon)(e^{\varepsilon\pi}-e^{-\varepsilon\pi})(\mu_1+\mu_2)},\quad \beta_{n1}=\alpha_{n2}$$

$$\beta_{n2}=\alpha_{n2}-M_n\frac{\varkappa_2+n+1/2+i\varepsilon}{n+3/2+i\varepsilon},\quad \alpha_{n1}=\alpha_{n2}-M_n\frac{\varkappa_1+n+1/2+i\varepsilon}{n+3/2+i\varepsilon}$$

Hence, we have obtained a sequence of singular solutions of problem (3.1)–(3.3) in which the displacements $u_{n2}^{(k)}(r,\varphi)=u_{\tan}^{(k)}$ have singularities of order $r^{n+3/2+i\varepsilon}$ ($n=0,1,2,\ldots$). Here, for convenience, we have considered complex-valued solutions, but clearly if we take their real or imaginary parts we can obtain real solutions with the orders of the singularities indicated.

It follows from the results obtained that the expansions of the displacements in the vicinity of the smooth crack contour have the form

$$u_{\tan}(r,\varphi,\tau)=\sum r^{n+1/2}f_n^*(\varphi,\tau)+\sum \operatorname{Re}(r^{n+3/2+i\varepsilon}g_n^*(\varphi,\tau))+\sum r^nh_n^*(\varphi,\tau)$$

$$u_3(r,\varphi,\tau)+iu_{\rm nor}(r,\varphi,\tau)=\sum r^{n+1/2+i\varepsilon}f_n(\varphi,\tau)+i\sum r^{n+3/2}g_n(\varphi,\tau)+\sum r^nh_n(\varphi,\tau)$$

Here τ is a parameter which defines the position of a point on the crack contour and (r,φ) are the polar coordinates in the plane passing through a given point on the contour, taken as the origin of coordinates, and orthogonal to ∂G. The functions $f_n^*(\varphi,\tau)$, $h_n^*(\varphi,\tau)$, $g_n(\varphi,\tau)$ are real-valued, while the functions $g_n^*(\varphi,\tau)$, $f_n(\varphi,\tau)$ and $h_n(\varphi,\tau)$ are complex-valued.

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